Local Cohomology

Lefschetz:

Thm $\left.\begin{array}{l} X \text{ proj vty} /_{k=\bar{k}} \\ H \subseteq X \text{ ample divisor} \\ \dim(X) \geq 2 \end{array}\right\} \Rightarrow \begin{array}{l} H \text{ connected } \& \\ \pi_1(H) \twoheadrightarrow \pi_1(X). \end{array}$

Thm Let U be the punctured Spec of a complete Noeth. local domain A,

$\left.\begin{array}{l} f \in m_A \\ \dim(A) \geq 3 \end{array}\right\} \Rightarrow \begin{array}{l} U \cap V(f) \text{ is connected } \& \\ \pi_1(U \cap V(f)) \twoheadrightarrow \pi_1(U). \end{array}$

Relationship: Take cone over X.

Strategy: Try to show $\Gamma(U, \mathcal{O}_U) \xrightarrow{\cong} \lim_n \Gamma(U, \overbrace{\mathcal{O}_U/f^n\mathcal{O}_U}^{\text{supported on } U \cap V(f)})$

so if $U \cap V(f)$ is NOT conn'd, $\exists$ idempotent in $\Gamma(U, \mathcal{O}_U)$.

Grothendieck: Show $\bigoplus_n \Gamma(U, f^n\mathcal{O}_U)$ & $\bigoplus_n H^1(U, f^n\mathcal{O}_U)$ are finite modules over the Rees alg. $\bigoplus_{n \geq 0} (f^n) = A \oplus (f) \oplus (f^2) \oplus \cdots$

Then you get the isom., which requires

$\begin{cases} \operatorname{depth}(A_{\mathfrak{p}}) \geq 2 \quad \text{for } \mathfrak{p} \subseteq m \text{ and } \dim(A/\mathfrak{p}) \geq 1 \\ f \quad \text{nonzero divisor.} \end{cases}$

(SGA 2, page 106).

Derived Completion.

- The derived cat of a ring A.

$D(A) = S^{-1}K(\mathrm{Mod}_A)$. $S = \mathrm{qis}$, $K(\mathrm{Mod}_A) =$ homtpy cat of $C(\mathrm{Mod}_A)$

· $- \overset{L}{\underset{A}{\otimes}} - : D(A) \times D(A) \longrightarrow D(A)$

$H^n(K \overset{L}{\underset{A}{\otimes}} M) = \mathrm{Tor}^A_{-n}(K, M)$.

· $R\mathrm{Hom}_A(-,-) : D(A)^{op} \times D(A) \longrightarrow D(A)$

$H^n(R\mathrm{Hom}_A(K,M)) = \mathrm{Hom}_{D(A)}(K, M[n])$
$= \mathrm{Ext}^n_A(K, M)$.

· $R\mathrm{Hom}_A(K, R\mathrm{Hom}_A(L, M)) = R\mathrm{Hom}_A(K \overset{L}{\underset{A}{\otimes}} \cancel{A}\, L, M)$.

- Rlim and derived limits.

If $(K_n)_{n\geq 1}$ is an inverse system in $D(A)$.

$\cdots \longrightarrow K_3 \longrightarrow K_2 \longrightarrow K_1$.

then there is an obj. $R\lim K_n \in D(A)$

$$\begin{tikzcd} & & R\lim K_n \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] & \\ \cdots \arrow[r] & K_3 \arrow[r] & K_2 \arrow[r] & K_1 \end{tikzcd}$$

Moreover, we have a short exact sequence:

$$0 \longrightarrow R^1\lim H^{i-1}(K_n) \longrightarrow H^i(R\lim K_n) \longrightarrow \lim H^i(K_n) \longrightarrow 0.$$

construction: make distinguished triangle

$$R\lim(K_n)_n \longrightarrow \prod K_n \longrightarrow \prod K_n \longrightarrow R\lim[1]$$
$$(\xi_n)_n \longmapsto (\xi_n - \mathrm{Im}(\xi_{n+1}))_n.$$

Fact $\prod K_n$ = take representatives $K_n^\cdot$ of K_n and take $\prod K_n^\cdot$.

Fact If $K^\cdot_{n+1} \longrightarrow K^\cdot_n$ s.t. $(K_n^i)_{n\geq 1}$ has ML. property for all i, then $R\lim(K_n) = \lim(K_n^\cdot)$ (naïve limits termwise).

e.g. $A = \mathbb{Z}$, $R\lim(\cdots \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z})$.

$$\begin{tikzcd} \cdots \arrow[r, "\cdot 2"] & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z} \arrow[d] \\ \cdots \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z}[1/2] \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z}[1/2] \arrow[d] \\ \cdots \arrow[r, "\cdot 2"] & \mathbb{Q}_2/\mathbb{Z}_2 \arrow[r] & \mathbb{Q}_2/\mathbb{Z}_2 \end{tikzcd}$$

$$R\lim(\cdots \xrightarrow{\cdot 2} \mathbb{Z}) = (\mathbb{Z}[1/2] \longrightarrow \mathbb{Q}_2) = (\mathbb{Z}_2/\mathbb{Z})[-1].$$

- Derived completion (Greenlees + ⋯).

Defn/Prop. Say $K \in D(A)$ is derived complete w.r.t. a f.g. ideal I iff TFEC.A.S. for all $f \in I$

① $\mathrm{Ext}^n_A(A_f, K) = 0 \quad \forall\, n \in \mathbb{Z}$.

② $R\lim(\cdots \longrightarrow K \xrightarrow{f} K \xrightarrow{f} K) = 0$.

③ $R\lim(K \overset{L}{\underset{A}{\otimes}} (A \xrightarrow{f^n} A)) \xrightarrow{\simeq} K$.

④ each $H^i(K)$ is derived complete in the sense of ①.

$$D(A) \supseteq \left(\begin{matrix}\text{derived} \\ \text{cplete guys}\end{matrix}\right) = \langle A_f, f \in I\rangle^{\perp} = D_{\mathcal{C}}(A)$$

$$D_{\mathcal{C}}(A) := \{K \in D(A) \mid H^i(K) \in \mathcal{C} \ \forall\, i \in \mathbb{Z}\}.$$

e.g. $\overset{I=(p)}{A=\mathbb{Z}_p}$, then $K=\mathbb{Z}_p$ satisfies ②.

Now we want to verify ①:

$$\mathrm{Ext}^i_{\mathbb{Z}_p}\left(\mathbb{Z}_p[\tfrac{1}{p}]=\mathbb{Q}_p,\ \mathbb{Z}_p\right)=0 \quad \forall i.$$

- $\mathbb{Z}_p$ has proj dim 1,
- $\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p,\mathbb{Z}_p)=0$
- $0\to\bigoplus_{\mathbb{N}}\mathbb{Z}_p \longrightarrow \bigoplus_{\mathbb{N}}\mathbb{Z}_p \xrightarrow{(1,1/p,1/p^2,\dots)} \mathbb{Q}_p \longrightarrow 0.$

  $e_i \longmapsto f_i - p f_{i+1}$

One verifies directly from that $\mathrm{Ext}^1_{\mathbb{Z}_p}(\mathbb{Q}_p,\mathbb{Z}_p)=0$.

Suppose $I=(f_1,\dots,f_r)$, the alternating Čech complex is

$$\overset{0}{A}\longrightarrow \overset{1}{\prod A_{f_i}}\longrightarrow\cdots\longrightarrow A_{f_1\cdots f_r}\longrightarrow 0$$

$$=\operatorname*{colim}_n \underbrace{\tilde{K}^\cdot(A,f_1^n,\dots,f_r^n)}_{\text{shifted Koszul cplx}}$$

$$R\mathrm{Hom}_A\left(\tilde{K}^\cdot(A,f_1^n,\dots,f_r^n),A\right)=K^\cdot(A,f_1^n,\dots,f_r^n) \leftarrow \deg -r,\dots,0$$

Thm The inclusion functor $D_{\mathcal{C}}(A)\longrightarrow D(A)$ has a left adjoint, called derived completion,

$$K\longmapsto K^\wedge = R\mathrm{Hom}_A\left((A\to\prod A_{f_i}\to\cdots\to A_{f_1\cdots f_r}),K\right)$$

$$=R\mathrm{Hom}_A\left(\operatorname*{colim}_n \tilde{K}^\cdot(A,f_1^n,\dots,f_r^n),K\right).$$

$$=\operatorname*{R\lim}_n R\mathrm{Hom}\left(\tilde{K}^\cdot(A,f_1^n,\dots,f_r^n),K\right)$$

$$=\operatorname*{R\lim}_n K\otimes^{\mathbb{L}}_A K^\cdot(A,f_1^n,\dots,f_r^n).$$

Thm. If A is Noetherian, then the pro-system $\{K(A,f_1^n,\dots,f_r^n)\}_{n\ge1}$ and $\{A/I^n\}_{n\ge1}$ are isom. in pro-$D(A)$.

Cor. If A is Noetherian, $K^\wedge_I = R\lim K\otimes^{\mathbb{L}}_A (A/I^n)$.

Ex. A, Noetherian ring, M, flat A-mod, then

$$M^\wedge\cong\hat{M}:=\varprojlim M/I^nM.$$

Lemma If A Noeth., M finite A-mod, then $M^\wedge\cong\hat{M}$ (Artin-Rees).

Ex. A is a general ring, $I=(f)$, M is a module.

$$H^i(M^\wedge)=\begin{cases}0 & i\neq 0,1\\ \lim M[f^n] = \text{Tate module of } M \text{ w.r.t. } f \ (=T_fM) & i=-1\\ 0\to R^1\lim M[f^n]\to H^0(M^\wedge)\to\hat{M}\to 0 & i=0.\end{cases}$$

Cor. If M is derived complete (w.r.t. I) $\Rightarrow$ $M\twoheadrightarrow\hat{M}$ (holds for all I).

Rmk. It follows from defn that $M^\wedge\xrightarrow{\cong}M$ for any derived cplt M.

$$\begin{array}{ccc} U & & \mathcal{F} \in Coh(\mathcal{O}_U) \\ \downarrow & & \\ \operatorname{Spec} A & I \subseteq A \rightsquigarrow I\cdot\mathcal{O}_U & \end{array}$$

Back to Grothendieck:

$$\mathcal{F}^\wedge = R\lim\left(\mathcal{F} \overset{\mathbb{L}}{\underset{\mathcal{O}_U}{\otimes}} \left(\mathcal{O}_U/I^n\mathcal{O}_U\right)\right) \cong \lim \mathcal{F}/I^n\mathcal{F} = R\lim \mathcal{F}/I^n\mathcal{F}.$$

$$D(A) \ni R\Gamma(U,\mathcal{F})^\wedge = R\Gamma(U,\mathcal{F}^\wedge) = R\Gamma(U, R\lim \mathcal{F}/I^n\mathcal{F}).$$
$$= R\lim R\Gamma(U, \mathcal{F}/I^n\mathcal{F})$$

Hence $0 \to R^1\lim H^{i-1}(U, \mathcal{F}/I^n\mathcal{F}) \to H^i(R\Gamma(U,\mathcal{F})^\wedge)$
$\to \lim H^i(U, \mathcal{F}/I^n\mathcal{F}) \to 0.$

also: $E_2^{a,b} = H^a(H^b(U,\mathcal{F})^\wedge) \Rightarrow H^{a+b}(R\Gamma(U,\mathcal{F})^\wedge).$

Grothendieck: $\left.\begin{array}{l} I = (f) \Longrightarrow \quad a \in \{0,-1\}. \\ H^0(U,\mathcal{F}) \,\&\, H^1(U,\mathcal{F}) \text{ finite} \end{array}\right\} \Rightarrow H^0(R\Gamma(U,\mathcal{F})^\wedge) \cong \widehat{H^0(U,\mathcal{F})}.$

Hence $\widehat{H^0(U,\mathcal{F})} \xrightarrow{\cong} \lim H^0(U, \mathcal{F}/I^n\mathcal{F}).$

Local Cohomology.

A is a ring, $I \subseteq A$ f.g. ideal. For any A-mod. M we set

$$M[I^n] = \{x \in M \mid fx = 0 \quad \forall f \in I^n\} = \operatorname{Hom}_A(A/I^n, M).$$
$$H^0_I(M) = M[I^\infty] = \bigcup_n M[I^n] = \operatorname{colim}_n \operatorname{Hom}_A(A/I^n, M).$$

Thm The right derived functor
$$R\Gamma_I = RH^0_I : D(A) \longrightarrow D(I^\infty\text{-torsion} / I\text{-power torsion modules})$$
is the right adjoint to the natural functor
$$D(I^\infty\text{-torsion}) \longrightarrow D(A).$$
Moreover, $H^i_I(K) := H^i(R\Gamma_I(K)) = \operatorname{colim}_n \operatorname{Ext}^i_A(A/I^n, K).$

Say $I = (f_1, \dots, f_r)$ and set $Z = V(I) \subseteq \operatorname{Spec}(A)$.
The alternating Čech cplx
$$C = (A \longrightarrow \prod_i A_{f_i} \longrightarrow \prod_{i<j} A_{f_if_j} \longrightarrow \cdots \longrightarrow A_{f_1\cdots f_r}).$$

Rmk $C_\mathfrak{p}$ is acyclic (homotopic to 0) for $\mathfrak{p} \notin Z$.

The cohomology modules of C are I^∞-torsion and the same is true for $K \overset{\mathbb{L}}{\underset{A}{\otimes}} C$ for any $K \in D(A)$.

Thm The functor $R\Gamma_Z : D(A) \longrightarrow D_{I^\infty\text{-torsion}}(A) := \left\{\begin{array}{l} K \in D(A) \text{ w/} \\ H^n(K) \in \{I^\infty\text{-torsion}\} \end{array}\right\}$
$$K \longmapsto K \overset{\mathbb{L}}{\underset{A}{\otimes}} C.$$
is right adjoint to the inclusion functor
$$D_{I^\infty\text{-torsion}}(A) \longrightarrow D(A).$$

Rmk $D_{I^\infty\text{-torsion}}(A)$ is ${}^\perp\langle M ;\ A_f \text{ module}, f\in I\rangle$.

Q: When is $D(I^\infty\text{-torsion}) \longrightarrow D_{I^\infty\text{-torsion}}(A)$ as equivalence.

($\mathcal{B}\subseteq\mathcal{A}$ Serre subcat., $\mathbb{D}(\mathcal{B})\longrightarrow\mathbb{D}^{\cdot}_{\mathcal{B}}(\mathcal{A})$ always exact functor)

Thm If A is Noetherian, then it is an equivalence and so we have $H^i_Z(K) = H^i(R\Gamma_Z(K)) = \operatorname{colim}_n \operatorname{Ext}^i_A(A/I^n, K^\cdot)$.

pf. use the structure of injective modules / Noeth. rings. They are direct sums of inj. hulls of residue fields; or use $\{K^\cdot(A, f_1^n, \cdots, f_r^n)\}_{n\geq 1}$ and $\{A/I^n\}_{n\geq 1}$ are isom. in pro-$D(A)$.

Comparison w/ cohomology.

$X = \operatorname{Spec}(A) \supseteq U = X\setminus Z = X\setminus V(I)$.

$K\in D(A) \rightsquigarrow \widetilde{K}\in D_{QCoh}(\mathcal{O}_X)$.

Thm There is a distinguished triangle

$$R\Gamma_Z(K)\longrightarrow K\longrightarrow R\Gamma(U, \widetilde{K}|_U)\longrightarrow R\Gamma_Z(K)[1]$$

in $D(A)$.

pf. $(0\longrightarrow \prod A_{f_i}\longrightarrow \prod A_{f_if_j}\longrightarrow\cdots\longrightarrow A_{f_1\cdots f_r})\longrightarrow C\longrightarrow A[0]$ triangle & $\nearrow$ $[-1]$ compute coh. $R\Gamma(U,-)$ universally.

Cor. M is an A-module and $\mathcal{F}=\widetilde{M}$ on X then we have an exact sequence

$$0\longrightarrow H^0_Z(M)\longrightarrow M\longrightarrow H^0(U,\mathcal{F})\longrightarrow H^1_Z(M)\longrightarrow 0$$

and isom. $H^i(U,\mathcal{F}) \xrightarrow{\sim} H^{i+1}_Z(M)$ for $i\geq 1$.

Cor. If A is Noeth. Then $H^i(U,\mathcal{F}) = \operatorname{colim}_n \operatorname{Ext}^{i+1}_A(A/I^n, M)$ $\forall i\geq 1$.

e.g./Rmk $\mathcal{F}$ coh. sheaf on $\mathbb{P}^n_k$. Then $\mathcal{F}$ corresponds to a finite graded $k[T_0,\cdots,T_n]$-mod. M.

$$\bigoplus_{e\in\mathbb{Z}} H^i(\mathbb{P}^n, \mathcal{F}(e)) \cong H^i(\operatorname{Spec}(k[T_0,\cdots,T_n])\setminus\{0\}, \widetilde{M}) \underset{i\geq 1}{\cong} H^{i+1}_{\{0\}}(M).$$

$X = \mathbb{A}^{n+1}_k \hookleftarrow \mathbb{A}^{n+1}_k\setminus\{0\} = U \xrightarrow{\pi} \mathbb{P}^n_k$. $\quad \pi_*\pi^*\mathcal{F} = \oplus\mathcal{F}(e)$, $\quad \widetilde{M}|_U = \pi^*\mathcal{F}$.

example If $I=(f_1,\cdots,f_r)$ generated by reg. sequence $f_1,\cdots,f_r$. then

$$H^m_Z(A) = \begin{cases} 0 & m\neq r \\ A[\frac{1}{f_1\cdots f_r}]/\sum A[\frac{1}{f_1\cdots \hat{f_i}\cdots f_r}] & m=r.\end{cases}$$

$$H^m_Z(M) = \operatorname{Tor}^A_{r-m}(M, H^r_Z(A)).$$

Lemma (characterization of depth) $(A,\mathfrak{m})$ Noeth. local ring. Let $d\geq 0$. Let M be a finite A-module. TFAE:

- $\operatorname{depth}(M)\geq d$
- $\operatorname{Ext}^i_A(A/\mathfrak{m}, M) = 0$, $i<d$
- $H^i_{\{\mathfrak{m}\}}(M) = 0$, $i<d$.

(Tag 0AVZ)

(Hartshome)

Lemma (A, m) Noeth. local.

this condition is preserved under completion and henselization $\longrightarrow$ $\operatorname{depth}(A) \geq 2 \Longrightarrow$ punctured Spec U is connected.

pf: If not, then $H^0(U, \mathcal{O}_U)$ has a nontrivial idempotent

Hence $A \longrightarrow H^0(U, \mathcal{O}_U)$ not an isom.

So $H^0_m(A) \neq 0$ or $H^1_m(A) \neq 0$.

Torsion v.s. Complete

Thm A is a ring, $I \subseteq A$ f.g. ideal. Then

$$D_{\{\text{derived cplt}\}}(A) \xleftrightarrow{\text{equivalent}} D_{I^\infty\text{-torsion}}(A).$$

$$K \longmapsto R\Gamma_Z(K)$$

$$K^\wedge \longmapsfrom K$$

pf. It suffices to show $\forall\, K \in D(A)$, that

$$R\Gamma_Z(K^\wedge) \xleftrightarrow{\text{qis}} R\Gamma_Z(K) \text{ and } R\Gamma_Z(K)^\wedge \xrightarrow{\text{qis}} K^\wedge.$$

triangle: $K \longrightarrow K^\wedge \longrightarrow R\mathrm{Hom}((\prod A_{f_i} \longrightarrow \prod A_{f_if_j} \longrightarrow \cdots \longrightarrow A_{f_1\cdots f_r}), K)$

apply $\overset{L}{\underset{A}{\otimes}} C$ to $\nearrow$ yields zero since they are in $\langle A_f\text{-mod}, f \in I\rangle$

triangle: $R\Gamma_Z(K) \longrightarrow K \longrightarrow K \overset{L}{\underset{A}{\otimes}} (\prod A_{f_i} \longrightarrow \prod A_{f_if_j} \longrightarrow \cdots \longrightarrow A_{f_1\cdots f_r})$

Similarly, apply $R\mathrm{Hom}_A(C, -)$ to $\nearrow$ yields zero.

Cor. For $K, L \in D(A)$ we have

$$R\mathrm{Hom}_A(K^\wedge, L^\wedge) \cong R\mathrm{Hom}_A(R\Gamma_Z(K), R\Gamma_Z(L)) \text{ in } D(A).$$

Example: $A = \mathbb{Z}_p$ and $I = (p)$. use $0 \to H^0_Z(M) \to M \to H^0(U, \mathcal{F}) \to H^1_Z(M) \to 0$

complete	Torsion
$\mathbb{Z}_p[0]$	$\mathbb{Q}_p/\mathbb{Z}_p[-1]$
$\mathbb{Z}/p^n\mathbb{Z}[0]$	$\mathbb{Z}/p^n\mathbb{Z}[0]$
$(\bigoplus_{n\geq 1} \mathbb{Z}_p)^\wedge[1]$	$(\bigoplus_{n\geq 1} \mathbb{Q}_p/\mathbb{Z}_p)[0]$

Example. $A = k[[x,y]]$, $I = m = (x,y)$

$A[2] \longleftarrow$ normalized dualizing cplx $\omega_A^\bullet$ for A.

$\updownarrow$ $(A[\tfrac{1}{xy}]/A[\tfrac{1}{x}] + A[\tfrac{1}{y}])[0] = E$ inj. hull of residue field.

$$R\mathrm{Hom}_A(M, \omega_A^\bullet) \cong R\mathrm{Hom}(R\Gamma_{\{m\}} M, E).$$

finite A-mod. hence complete.

Hence $\mathrm{Ext}^{-i}_A(M, \omega_A^\bullet) \cong \mathrm{Hom}(H^i_{\{m\}}(M), E)$.

Example. In general $D(I^\infty\text{-torsion}) \longrightarrow D_{I^\infty\text{-torsion}}(A)$ is not full.

Take $A = \mathbb{Z}[f, x, x_n]/(fx, x_n - f x_{n+1})$, $I = (f)$.

$$\begin{tikzcd} \mathrm{Ext}^2_{D_{I^\infty\text{-torsion}}(A)}(A/f, A[f]) \ni [0 \arrow[r] & A[f] \arrow[r] \arrow[d, "\cong"] & A \arrow[r, "\cdot f"] \arrow[d, "\varphi"] & A \arrow[r] \arrow[d, "\psi"] & A/fA \arrow[r] \arrow[d, "\cong"] & 0] \\ 0 \arrow[r] & A[f] \arrow[r] & M \arrow[r] & N \arrow[r] & A/fA \arrow[r] & 0 \end{tikzcd}$$

is not in the image of $\mathrm{Ext}^2_{D(I^\infty\text{-torsion})}(A/f, A[f])$ | otherwise $\exists\, \varphi, \psi \ldots$

Replace M, N by $\mathrm{Im}(\varphi)$, $\mathrm{Im}(\psi)$. We see that we may assume M is cyclic. Hence $f^n M = 0$ for some n.

This ~~cond~~ contradicts $A[f] \hookrightarrow M$ & $x_{n+1} \in A[f]$ has $f^n \cdot x_{n+1} = x_1 \neq 0$.

Dualizing complexes

Defn. A is a Noeth. ring. Let $D_{coh}(A) = \left\{ \begin{array}{l} K^\cdot \in D(A), \text{ s.t. } \quad \forall i \\ H^i(K^\cdot) \text{ are finite } A\text{-mod} \end{array} \right\}$

$\omega_A^\cdot \in D(A)$ is <u>dualizing cplx</u> if

(1) $\omega_A^\cdot$ has finite inj. dimension

(2) $\omega_A^\cdot \in D_{coh}(\mathbf{A})$ so $\omega_A^\cdot \in D^b_{coh}(\mathbf{A})$.

(3) $A \longrightarrow RHom(\omega_A^\cdot, \omega_A^\cdot)$ is an isom in $D(A)$.

Lemma. If $\omega_A^\cdot$ is a dualizing cplx, then

$D(K) = RHom(K, \omega_A^\cdot)$ gives anti-equivalence of

$$D^b_{coh}(A) \longleftrightarrow D^b_{coh}(A)$$

$$D^+_{coh}(A) \longleftrightarrow D^-_{coh}(A)$$

$$D_{coh}(A) \longleftrightarrow D_{coh}(A).$$

Defn. Say $L \in D(A)$ is invertible iff $\forall \mathfrak{p} \in \mathrm{Spec}(A)$, $\exists f \in A \setminus \mathfrak{p}$ s.t. $L \otimes^{\mathbb{L}}_A A_f \cong A_f[n]$ some $n \in \mathbb{Z}$.

Lemma. If $\omega_A^\cdot$, $\omega_A'^\cdot$ are dualizing cplxes, then

$L = RHom_A(\omega_A^\cdot, \omega_A'^\cdot)$ is invertible and

$$\omega_A^\cdot \otimes^{\mathbb{L}}_A L \xrightarrow{\cong} \omega_A'^\cdot.$$

sketch pf: (for A, local) (A, m, κ). Note that $F = D' \circ D : D^b_{coh}(A) \longrightarrow D^b_{coh}(A)$ an A-linear eq.

The only "simple obj." in $D^b_{coh}(A)$ are κ shifted.

$\Rightarrow F(\kappa) = \kappa[n]$ for a unique $n \in \mathbb{Z}$

$$\mathrm{Ext}^i_A(F(A), \kappa[n]) = \mathrm{Ext}^i_A(A, \kappa) \cong \begin{cases} 0 & i \neq 0 \\ \kappa & i = 0. \end{cases}$$

$$\subseteq \mathrm{Ext}^i_\kappa(F(A) \otimes^{\mathbb{L}}_A \kappa, \kappa[n]) \Longrightarrow F(A) \otimes^{\mathbb{L}}_A \kappa \cong \kappa[n]$$

$$\xrightarrow[\text{minimal res. of } F(A)]{\text{look at}} F(A) \cong A[n].$$

Facts: Let $\omega_A^\cdot$ be a dualizing cplx

- $S^{-1}\omega_A^\cdot$ is dualizing cplx for $S^{-1}A$.
- $A \longrightarrow B$ finite, $RHom_A(B, \omega_A^\cdot)$ dualizing for B.
- $\omega_A^\cdot \otimes^{\mathbb{L}}_A A[x]$ dualizing for $A[x]$.
- If $A = \kappa$, then $\kappa[0]$ is a dualizing cplx or $A = \mathbb{Z}$, then $\omega_A^\cdot = \mathbb{Z}[1] \simeq (\underset{-1}{\mathbb{Q}} \longrightarrow \underset{0}{\mathbb{Q}/\mathbb{Z}})$.

Defn (A, m, κ) Noeth. local ring, we say $\omega_A^\cdot$ is <u>normalized</u> iff $RHom_A(\kappa, \omega_A^\cdot) \cong \kappa[0]$.

e.g. $A = \kappa[[x]]$, then $\omega_A^\cdot = A[1]$ normalized dualizing cplx.

Matlis duality: Artinian case.

(A, m, κ) artinian local ring.

$\kappa \subseteq E$ inj. hull. (c.f. [Tag 08Y1, SP]). & Tag 08Z1

e.g. $A = k[x,y]/(x^2, x\cancel{y}, y^2)$
$\begin{tikzcd} & 1 \arrow[dl, "x"'] \arrow[dr, "y"] & \\ x & & y \end{tikzcd}$
$\begin{tikzcd} e \arrow[dr, "x"'] & & f \arrow[dl, "y"] \\ & g & \end{tikzcd}$
$k = k\cdot(xe) = k\cdot(yf)$
$\mathbf{E} = \mathrm{Hom}_k(A, k) = \dfrac{Ae \oplus Af}{\langle xe - yf, ye, xf \rangle}$.

Fact/Defn $\kappa \subseteq E$ inj. hull means $\forall$ nontrivial $M \subseteq E$, $M \cap \kappa \neq \{0\}$.

$\mathrm{Hom}_A(\kappa, E) \cong \kappa$.

Claim: The functor $\mathrm{Hom}_A(-, E) : \mathrm{Mod}_A^{f.g.} \longrightarrow \mathrm{Mod}_A^{f.g.}$ is an anti-equiv. w/ inverse itself.

Rmk: $\omega_A^\cdot = E[0]$ is the normalized duality cplx for A.

pf Claim: first show $\ell(\mathrm{Hom}_A(M, E)) = \ell(M)$.

then show $ev: M \longrightarrow \mathrm{Hom}_A(\mathrm{Hom}_A(M, E), E)$ inj.

Rmk actually in ~~this~~ the case of $k \hookrightarrow A \twoheadrightarrow k$, we have

$\mathrm{Hom}_A(M, E) = \mathrm{Hom}_k(M, k) \ldots$

Matlis duality: general case.

$(A, \mathfrak{m}, \kappa)$ local Noeth. $\kappa \subseteq E$ inj. hull.

Claim If A is $\mathfrak{m}$-adically complete. Then the functor $\mathrm{Hom}_A(-, E)$ induces an anti-eq. $\{A\text{-modules w/ a.c.c. (finite } A\text{-mod)}\}$

$$\updownarrow$$

$\{A\text{-mod. w/ d.c.c.}\}$

with inverse itself.

Idea: Show $E = \bigcup_n \underbrace{E[\mathfrak{m}^n]}_{\text{inj hull of } \kappa \text{ over } A/\mathfrak{m}^n}$.

Hence $\mathrm{End}(E) \cong \hat{A} \cong A$.

Then show E has d.c.c. and any A-mod. M w/ d.c.c. fits into exact sequence $0 \to M \to E^{\oplus r} \to E^{\oplus s}$.

e.g. $\mathbb{Z}_p$. $E = \mathbb{Q}_p/\mathbb{Z}_p$.

$A = k[\![x_1, \cdots, x_d]\!]$ $\quad E = A[\frac{1}{x_1 \cdots x_d}] / \sum A[\frac{1}{x_1 \cdots \hat{x_i} \cdots x_d}]$.

Back to $\omega_A^\cdot$:

Lemma $(A, \mathfrak{m}, \kappa)$ Noeth. local, E inj. hull of κ, $\omega_A^\cdot$ normalized dualizing cplx. Then:

$$\left.\begin{array}{l} R\mathrm{Hom}_A(-, \omega_A^\cdot) \\ \mathrm{Hom}_A(-, E) \end{array}\right\} : \begin{array}{c}\text{finite length} \\ A\text{-mod}\end{array} \longrightarrow \begin{array}{c}\text{finite length} \\ A\text{-mod}\end{array}$$

are isom. functor & induces anti-equiv.

pf. both are anti-eq. + show $\exists !$ self-equiv $\circlearrowright$ $\boxed{\text{finite length } A\text{-mod.}}$

Lemma. $R\Gamma_\mathfrak{m}(\omega_A^\cdot) \cong E[0]$ w/ notation as above.

pf. $H^i_\mathfrak{m}(\omega_A^\cdot) = \mathrm{colim}_n \mathrm{Ext}^i_A(A/\mathfrak{m}^n, \omega_A^\cdot)$

$$= \mathrm{colim}_n \begin{cases} 0 & i \neq 0 \\ \mathrm{Hom}_A(A/\mathfrak{m}^n, E) & i = 0 \end{cases} = \begin{cases} 0 & i \neq 0 \\ E & i = 0. \end{cases}$$

Grothendieck local duality thm.

Recall from last time: $\underset{\text{(Noeth.)}}{A} \supseteq \underset{\text{(f.g.)}}{I} \Rightarrow R\mathrm{Hom}_A(K^\wedge, M^\wedge) \cong R\mathrm{Hom}_A(R\Gamma_I(K), R\Gamma_I(M))$

$$\wr\!\wr$$

$R\mathrm{Hom}_A(\mathcal{L}, R\mathrm{Hom}_A(K, M)) \cong R\mathrm{Hom}_A(K \otimes^{\mathbb{L}}_A \mathcal{L}, M) \cong R\mathrm{Hom}_A(K, M^\wedge)$

$$\|$$

$R\mathrm{Hom}(K, M)^\wedge$

So: $R\mathrm{Hom}_A(K, \omega_A^\cdot)^\wedge \cong R\mathrm{Hom}_A(R\Gamma_\mathfrak{m}(K), E)$.

If M is a finite A-mod. $\mathrm{Ext}^{-i}_A(M, \omega_A^\cdot)^\wedge$ is usual completion (as $\mathrm{Ext}\cdots$ is finite) & $\cong \mathrm{Hom}_A(H^i_\mathfrak{m}(M), E)$

Cor. $H^i_\mathfrak{m}(M)$ has d.c.c. (as an $\hat{A}$-mod.?) $\leftarrow$ actually it's the same since $H^i_\mathfrak{m}(M)$ are $\mathfrak{m}^\infty$-torsion $\forall i$.

Example: $A = \mathbb{C}[\![x_1, \cdots, x_d]\!]$. $M = A$.

$$H^{-i}(\omega_A^\cdot) = \mathrm{Ext}^{-i}_A(A, \omega_A^\cdot) \cong \mathrm{Hom}_A(H^i_\mathfrak{m}(A), E)$$

actually $\omega_A^\cdot = A[d] \longleftrightarrow H^i_\mathfrak{m}(A) = \begin{cases} 0 & i \neq d \\ E & i = d. \end{cases}$

$M = A/fA$. $f \in \mathfrak{m}$, $f \neq 0$.

LHS $\mathrm{Ext}^{-i}_A(M, \omega_A^\cdot) = \mathrm{Ext}^{-i}_A(\underset{-1}{A} \xrightarrow{f} \underset{0}{A}, A[d])$

$$= \mathrm{Ext}^{d-i}_A(\underset{0}{A} \xrightarrow{\cdot f} \underset{1}{A}) = \begin{cases} 0 & i \neq d-1 \\ A/fA & i = d-1. \end{cases}$$

$$\Rightarrow H^i_\mathfrak{m}(A/fA) = \begin{cases} 0 & i \neq d-1 \\ \mathrm{Hom}_A(A/fA, E) = E[f] & i = d-1. \end{cases}$$

Grothendieck's finiteness thm   SGA 2 Exp VIII.

Let $j: U \hookrightarrow X$ be an open immersion of Noeth. schemes. $\mathcal{F}$ coh. on U, $X \setminus Z = U$.

Q: ⓐ when is $j_*\mathcal{F}$ coherent?
ⓑ when is $R^p j_* \mathcal{F}$ coherent for $p < s$?

Answer ⓐ by Kollar: $j_*\mathcal{F}$ coherent iff

$\forall u \in \mathrm{Ass}(\mathcal{F})$, $z \in \overline{\{u\}}$, $z \in Z$, $\forall \mathfrak{p} \in \mathrm{Ass}(\widehat{\mathcal{O}}_{\overline{\{u\}}, z})$

we have $\dim(\widehat{\mathcal{O}}_{\overline{\{u\}}, z}/\mathfrak{p}) \geq 2$.

Rmk If X is excellent or loc. has dualizing cplx, it suffices to require $\dim(\mathcal{O}_{\overline{\{u\}}, z}) \geq 2$, i.e. Z is codim'n ≥ 2 w.r.t. $\mathrm{Supp}(\mathcal{F})$...

Answer ⓑ by Grothendieck: Assume X locally has a dualizing cplx. Then given $n \geq 0$, $R^p j_*\mathcal{F}$ coherent for all $p < n$ $\Leftrightarrow$

$\forall u \in \mathrm{Supp}(\mathcal{F})$, $z \in \overline{\{u\}}$, $z \in Z$,

we have $\dim(\mathcal{O}_{\overline{\{u\}}, z}) + \mathrm{depth}(\mathcal{F}_u) > n$.

Relation to local coh.: We can always extend $\mathcal{F}$ to a ~~co~~ coh sheaf $\mathcal{G}$ on ~~X~~ $\overline{\mathrm{Supp}(\mathcal{F})}$ (closure). Then we have

$$H^i_Z(\mathcal{G}) \to H^i(X, \mathcal{G}) \to \underset{\substack{\| \\ H^i(U, \mathcal{F})}}{H^i(U, \mathcal{G})}.$$

Hence we need to understand $H^i_Z(M)$, like when is this finite ...

More general supports:

A, Noeth. ring, $T \subseteq \mathrm{Spec}(A)$ stable under specialization. So T is directed union of closed subset Z of $\mathrm{Spec}(A)$.

$$H^i_T(M) := \underset{\substack{Z \subseteq T \\ \text{closed}}}{\mathrm{colim}}\, H^i_Z(M).$$

There is also derived version $R\Gamma_T(M): D(A) \to D_T(A)$ (if $\dim A < \infty$)

Finiteness of local cohomology, following Faltings.

Lemma: A & T as above. M finite A-mod. $n \geq 0$. TFAE.

(1) $H^i_T(M)$ is finite A-mod for all $i \leq n$.

(2) $\exists$ ideal $J \subseteq A$, $V(J) \subseteq T$ which annihilates $H^i_T(M)$ for $i \leq n$.

pf. (1) $\Rightarrow$ (2) easy.

(2) $\Rightarrow$ (1): $n = 0$, both (1) & (2) are true.

$n > 0$, standard trick: $M' = M/H^0_T(M)$.

$$0 \to H^0_T(M) \to M \to M' \to 0.$$

conclude: $H^0_T(M') = 0$ & $H^i_T(M) \xrightarrow{\cong} H^i_T(M')$ $\forall$ $i \geq 1$.

Hence we may replace M by M', so may assume $\mathrm{Ass}(M) \cap T = \emptyset$. Let J be as in (2). Then by $\nearrow$, can find $f \in J$ not in any of $\mathrm{Ass}(M)$, hence a nonzero divisor of M.

$$0 \to M \xrightarrow{\cdot f} M \to M/fM \to 0.$$

Since $f \in J$, the long exact seq. breaks into

$$0 \to H^{i-1}_T(M) \to \underbrace{H^{i-1}_T(M/fM)}_{\text{killed by } J^2} \to H^i_T(M) \to 0.$$

By induction, done!

Faltings Annihilator Theorem.

Thm (Faltings). Assume A as before, has $\omega_A^\bullet$, $T \subseteq T' \subseteq \mathrm{Spec}(A)$, specialization stable subsets. M is a finite A-mod. $s \geq 0$, TFAE:

① $\exists\, J \subseteq A$ w/ $V(J) \subseteq T'$ s.t. J annihilates $H^i_T(M)$ $\forall\, i \leq s$.

② $\forall\, \mathfrak{p} \notin T'$, $\forall\, \mathfrak{q} \in T$, $\mathfrak{p} \subseteq \mathfrak{q}$, we have

$$\varphi_M(\mathfrak{p}, \mathfrak{q}) \overset{\text{def}}{=\!=} \mathrm{depth}(M_\mathfrak{p}) + \dim\left((A/\mathfrak{p})_\mathfrak{q}\right) > s.$$

Cor. In the situation above, $H^i_T(M)$ finite $\forall\, i \leq s \Leftrightarrow$

$$\forall\, \mathfrak{p} \subseteq \mathfrak{q},\ \mathfrak{p} \notin T,\ \mathfrak{q} \in T,\ \varphi_M(\mathfrak{p}, \mathfrak{q}) > s.$$

Preparation The dimension function $\delta : \mathrm{Spec}(A) \to \mathbb{Z}$ associated w/ $\omega_A^\bullet$ is defined by requirement that

$$(\omega_A^\bullet)_\mathfrak{p}[-\delta(\mathfrak{p})] = \text{normalized dualizing cplx of } A_\mathfrak{p}.$$

Let $E^i = \mathrm{Ext}^i_A(M, \omega_A^\bullet)$, $E^i_\mathfrak{p} \underset{\text{dual over } A_\mathfrak{p}}{\overset{\text{Matlis}}{\longleftrightarrow}} H^{-i-\delta(\mathfrak{p})}_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p})$.

These modules have the same annihilators in $A_\mathfrak{p}$.

pf of Thm ② ⇒ ①: By induction on s, we may assume $\exists\, J' \subseteq A$, w/ $V(J') \subseteq T'$ s.t. J' annihilates $H^i_T(M)$ $\forall\, i < s$.

Set $T_n = \{\mathfrak{p} \in T \mid \delta(\mathfrak{p}) \leq n\}$.

By decreasing induction on n, we'll find $J_n \subseteq A$, $V(J_n) \subseteq T'$ w/ $\mathrm{Ass}(J_n \cdot H^s_T(M)) \subseteq T_n$.

go to $n \ll 0$, done!

Assume we have J_n already, and $\mathfrak{q} \in T_n \setminus T_{n-1}$, i.e. $\delta(\mathfrak{q}) = n$.

We have $H^i_T(M)_\mathfrak{q} = H^i_{T_\mathfrak{q}}(M_\mathfrak{q})$ $\forall\, i$, where $T_\mathfrak{q} \subseteq \mathrm{Spec}(A_\mathfrak{q})$ is the inverse image of T.

Claim: $\exists\, J'' \subseteq A$, w/ $V(J'') \subseteq T'$, s.t. $\forall\, \mathfrak{q} \in T_n \setminus T_{n-1}$, the ideal J'' kills $H^s_\mathfrak{q}(M_\mathfrak{q})$

Now, granting Claim above, consider $H^a_{\mathfrak{q}A_\mathfrak{q}}(H^b_{T_\mathfrak{q}}(M_\mathfrak{q})) \Rightarrow H^{a+b}_{\mathfrak{q}A_\mathfrak{q}}(M_\mathfrak{q})$

for $b < s$, each term is killed by J'.

property of $J_n \Rightarrow$ $\underbrace{J_n \cdot H^s_{T_\mathfrak{q}}(M_\mathfrak{q})}_{\text{ass. primes have } \delta \leq n.} \subseteq H^0_\mathfrak{q}(H^s_{T_\mathfrak{q}}(M_\mathfrak{q}))$.

So only $\mathfrak{q}$ in ass. primes

The s.s. $\Rightarrow \mathrm{Ass}(J'' \cdot (J')^s \cdot J_n \cdot H^s_T(M)) \not\ni \mathfrak{q}$, $\forall\, \mathfrak{q} \in T_n \setminus T_{n-1}$.

By compactness of $T_n \setminus T_{n-1}$, we are done.

pf of Claim: J'' needs to annihilate $E^{-n-s}_\mathfrak{q}$ $\forall\, \mathfrak{q} \in T_n \setminus T_{n-1}$.

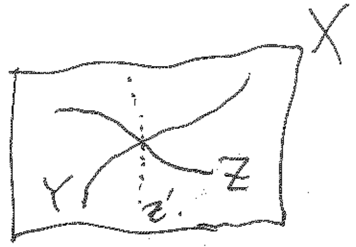
Since E^{-n-s} is finite, it suffices to do this one prime at a time.

We need to show $\mathrm{Supp}(E^{-n-s}) \cap \mathrm{Spec}(A_\mathfrak{q}) \subseteq T'$.

$\Leftrightarrow$ $\forall\, \mathfrak{p} \subseteq \mathfrak{q}$, $\mathfrak{p} \notin T'$, $(E^{-n-s})_\mathfrak{p} = 0$.

(preparation) $\underset{\text{duality}}{\overset{\text{local}}{\Longleftrightarrow}}$ $H^{n+s-\delta(\mathfrak{p})}_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ $\quad (\delta(\mathfrak{p}) - \delta(\mathfrak{q}) = \dim((A/\mathfrak{p})_\mathfrak{q})$.

$\underset{\text{of } \delta}{\overset{\text{defn}}{\Longleftrightarrow}}$ $H^{s-\dim((A/\mathfrak{p})_\mathfrak{q})}_{\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}) = 0 \Longleftarrow$ by $s - \dim((A/\mathfrak{p})_\mathfrak{q}) < \mathrm{depth}(M_\mathfrak{p})$ (②), Yeah!



Prolegomenon : Motivation

X Noeth. scheme. $Y, Z \subseteq X$ closed subschemes. $\mathcal{I} \subseteq \mathcal{O}_X$ ideal sheaf of Y. $U = X \setminus Z$. $\mathcal{F}$ coh. sheaf of $\mathcal{O}_U$-mod.
Y_n = n-th infinitesimal nbhd of $Y = V(\mathcal{I}^n)$

There're canonical maps

$$\begin{tikzcd} H^i(U,\mathcal{F}) \arrow[r, "(*)"] \arrow[d] & \lim\limits_n H^i(Y_n \cap U, \mathcal{F}/\mathcal{I}^n\mathcal{F}) \\ \operatorname*{colim}\limits_{\substack{Y_n \cap U \subseteq V \subseteq U \\ \text{open}}} H^i(V,\mathcal{F}) \arrow[ur, "(**)"'] & \end{tikzcd}$$

Question: What are natural conditions which imply $(*)$ or $(**)$ is an isom for $i \leq$ cutoff?

Rmk: X is not used yet in the formulation.

Projective case: $X = U$ = proj. vty, $Z = \phi$, Y = subvty. $H^i(X,\mathcal{F}) \to \lim\limits_n H^i(Y_n, \mathcal{F}/\mathcal{I}^n\mathcal{F})$.
Lefschetz type question. Taking cones, this reduces to the affine case.

Affine case: $X = \operatorname{Spec} A$, $Y = V(I)$, $Z = V(J)$, $\mathcal{F} = \widetilde{M}|_U$ where M is a f.g. A-mod.

Variant question (only in the affine case): When is $(*)$ or $(**)$ an isom. up to I-adic cpltion?

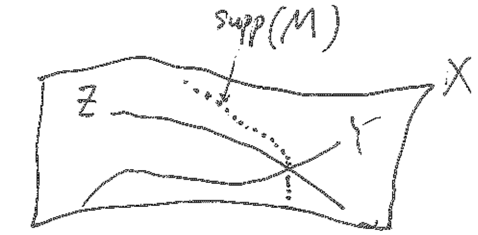
Example affine case: $Z = \phi$, $U = X$, $i = 0$.

$$\begin{tikzcd} M \arrow[r, "(*)"] \arrow[d] & \lim\limits_n M/I^nM \\ S^{-1}M \arrow[ur, "(**)"'] & \end{tikzcd}$$

$A = M$, A reg. $\longrightarrow S^{-1}M$

$S = \{f \in A \mid f \bmod I \text{ is a unit}\}$

variant question is OK
original question is OK if A is I-adically cplt.



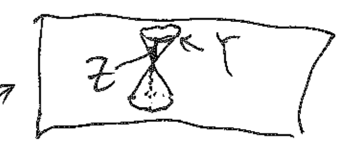
From now on, A is I-adically cplt.

Example Affine case: If $\operatorname{Supp}(M) \cap Y \subseteq Z \cap Y$.
Then

$$\begin{tikzcd} 0 \neq \Gamma(U,\mathcal{F}) \arrow[r, "(*)"] \arrow[d] & \lim\limits_n \Gamma(U \cap Y_n, \mathcal{F}/I^n\mathcal{F}) = 0 \\ \operatorname*{colim}\limits_{\substack{Y_n \cap U \subseteq V \subseteq U \\ \text{open}}} \Gamma(V,\mathcal{F}) = 0 \arrow[ur] & \end{tikzcd}$$

Conclusion $(**)$ looks better for original question.

Example Affine case: X regular, Y irreducible, $Z \subseteq Y$ codim'n 1 in Y. $\mathcal{F} = \mathcal{O}$, $M = A$
Then (1) $\lim \Gamma(U \cap Y_n, \mathcal{O})$ is often huge, but not always.
(2) $\operatorname*{colim}\limits_{Y_n \cap U \subseteq V \subseteq U} \Gamma(V, \mathcal{O}) = \begin{cases} A & \text{if } Z \text{ is not support of an eff. Cartier divisor on } Y. \\ \operatorname*{colim}\limits_{V(f) \cap Y = Z} A_f & \leftarrow \text{set theoretically.} \end{cases}$



Specific Example

$A = k[[x_1, x_2, x_3, x_4]]$. $Y = V(x_1x_2 - x_3x_4)$
$Z = V(x_1x_2 - x_3x_4, x_1x_4 - x_3^2, x_2x_3 - x_4^2) = \{(a, at^3, at, at^2)\}$.

Claim 1: There is no $f \in A$, s.t. $V(f) \cap Y = Z$ set theoretically.
Claim 2: $U \cap Y = Y \setminus Z$ is affine.
$\therefore$ The variant question is subtle as its answer depends not just on dimension & depth.

Reformulation using local cohomology.

Assume we are in the affine case with A: I-adically cplt + M finite.

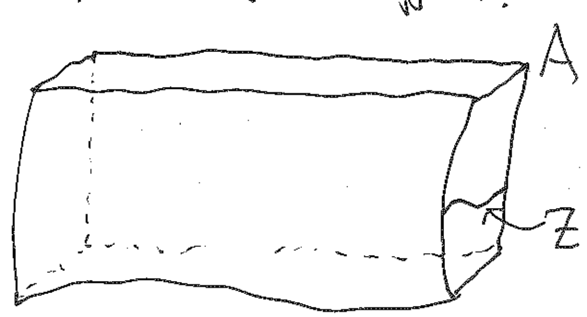
Then the questions are

$$\begin{array}{ccc} H^i_J(M) & \xrightarrow{(*)} & \lim\limits_n H^i_J(M/I^nM) \\ \downarrow & \nearrow_{(**)} & \\ \operatorname*{colim}\limits_{\substack{J' \subseteq J \\ V(J')\cap V(I) = V(J)\cap V(I)}} H^i_{J'}(M) & & \end{array}$$

for $i \le \text{cutoff} + 1$ are equiv. to original question.

Thm (to be proved) $I = (f) \subseteq J$, A = domain, $M = A$. every irred. component Z has codimension ≥ 2 in X.

Then $(**)$ isom for cutoff $= 1$.



related to questions in rigid geometry.

Completion of local cohomology.

Lemma. $I, J \subseteq A$, ideals in Noeth. ring, M – finite A-mod. Then

$$R\Gamma_J(M)^\wedge_I = R\lim R\Gamma_J(M/I^nM).$$

In particular, we get SES: $0 \to R^1\lim H^{i-1}_J(M/I^nM) \to H^i(R\Gamma_J(M)^\wedge_I) \to \lim H^i_J(M/I^nM) \to 0$.

~~Pf~~ pf: Say $J = (g_1, \cdots, g_m)$. $R\Gamma_J(M) = (M \to \prod M_{g_i} \to \cdots \to M_{g_1\cdots g_m})$.

Claim: $(M_g)^\wedge \cong \widehat{M_g}$ ← usual completion.

$$(M_g)^\wedge \cong (M \otimes_A A_g)^\wedge \overset{A_g\text{ flat}}{\cong} (M \overset{L}{\otimes}_A A_g)^\wedge \overset{M\text{ finite}}{\cong} M \overset{L}{\otimes}_A (A_g)^\wedge \overset{A_g\text{ flat}}{\cong} M \overset{L}{\otimes}_A \widehat{A_g}$$
$$\overset{\widehat{A_g}\text{ flat}}{\cong} M \otimes_A \widehat{A_g} \overset{M\text{ finite}}{\cong} \widehat{M \otimes_A A_g} \cong \widehat{M_g}.$$

$$R\Gamma_J(M)^\wedge_I = \mathrm{Tot}(M^\wedge_I \to \prod M_{g_i}{}^\wedge_I \to \cdots \to M_{g_1\cdots g_m}{}^\wedge_I).$$
$$= (\widehat{M} \to \prod \widehat{M_{g_i}} \to \cdots \to \widehat{M_{g_1\cdots g_m}}).$$
$$\overset{(M/IM)_g \,\cong\, M_g/IM_g}{\cong} R\lim(M/I^nM \to \prod (M/I^nM)_{g_i} \to \cdots \to (M/I^nM)_{g_1\cdots g_m})$$
$$= R\lim R\Gamma_J(M/I^nM).$$

Lemma. If A is I-adically cplt, then the map (M- finite)

$$\operatorname*{colim}_{\substack{J' \subseteq J \\ V(J')\cap V(I) = V(J)\cap V(I)}} H^0_{J'}(M) \xrightarrow{(**)} \lim H^0_J(M/I^nM) = H^0(R\Gamma_J(M)^\wedge_I)$$ is an isomorphism.

Pf. $\text{RHS} = \mathrm{Ker}(\widehat{M} \to \prod \widehat{M_{g_i}}) \overset{A\text{ is cplt w.r.t. }I}{=\!=} \mathrm{Ker}(M \to \prod \widehat{M_{g_i}})$

$$\mathrm{Ker}(M_{g_i} \to \widehat{M_{g_i}}) = \bigcap_n I^n M_{g_i} \underset{\text{Tag 00IQ}}{\overset{\text{Krull's intersection Thm}}{=\!=}} \{s \in M_{g_i}, \text{ s.t. } \operatorname{Supp}(s) \cap V(I) = \emptyset \text{ in } \operatorname{Spec}(A_{g_i})\}$$

$U = \mathrm{Spec}(A) \setminus V(J) = \bigcup \mathrm{Spec}(A_{g_i})$

$\Rightarrow \mathrm{Ker}(M \to \prod \widehat{M_{g_i}}) = \{s \in M,\ \mathrm{supp}(s) \cap U \cap V(I) = \phi\}$

Whereas $LHS = \{s \in M \mid (\mathrm{supp}(s) \cup V(J)) \cap V(I) = V(J) \cap V(I)\}$

Unraveling defn, it's equivalent to $\mathrm{Supp}(s) \cap V(I) \subseteq V(J)$.

which is equiv. to $\mathrm{Supp}(s) \cap V(I) \cap U = \phi$.

Lemma Suppose A (not necessarily cplt) is local and $V(J) \cap V(I) = \{m_A\}$

Then $\{H^i_J(M/I^nM)\}_{n \geq 1}$ has ML.

$\Rightarrow H^i(R\Gamma_J(M)^\wedge_I) = \lim H^i_J(M/I^nM)$.

pf. $H^i_J(M/I^nM) = H^i_m(M/I^nM)$ which is d.c.c. by previous lecture.

Cohomological dimension.

Lemma. A = Noeth. ring, $I \subseteq A$ ideal, $d \geq -1$. TFAE:

① $H^i_I(M) = 0$, $\forall i > d$, all A-mod. M.

② $H^i_I(A) = 0$, $\forall i > d$.

③ if $d=-1$, then $V(I) = \phi$, if $d=0$, then $V(I)$ open, and if $d \geq 1$, then $H^i(\mathrm{Spec}(A) \setminus V(I), \mathcal{F}) = 0, \forall i > d-1$, any q-coh. $\mathcal{F}$.

Defn. we say $cd(A, I)$ = cohomological dimension of I in A = smallest integer d that the lemma above holds.

Lemma (Serre's criterion) $cd(A,I) \leq 1 \Leftrightarrow \mathrm{Spec}(A) \setminus V(I)$ affine.

Lemma. If $V(I) = V(f_1, \cdots, f_r) \Rightarrow cd(A, I) \leq r$.

pf. $\mathrm{Spec}(A) \setminus V(I)$ has ~~aff.~~ a cov. by r affine charts.

Lemma. (A, m) local, then $cd(A, m) = \dim(A)$

pf. ~~Say~~ First show $cd(A,m) \leq d$ b/c $\exists$ system of parameters of length d.

Completing A to achieve that $\exists\ \omega_A^\bullet$, normalized dualizing cplx and $d = \dim(A)$. Then local duality gives that

$H^d_m(A) \xleftrightarrow[\text{duality}]{\text{Matlis}} H^{-d}(\omega_A^\bullet) = \omega_A$ the dualizing module of A and its support are the components of $\mathrm{Spec}(A)$ of dim'n d.

Lemma $A \to B$ ring map, then $cd(B, IB) \leq cd(A, I)$

pf. $\mathrm{Spec}(B) \setminus V(IB) \to \mathrm{Spec}(A) \setminus V(I)$ is an affine map.

Cor. $\mathfrak{p} \subseteq A$ prime, $\mathfrak{q} \subseteq V(\mathfrak{p}) \cap V(I)$ generic pt of the intersection. Then $\dim((A/\mathfrak{p})_\mathfrak{q}) \leq cd(A, I)$. "codim'n estimate"

pf.
$$\begin{array}{ccc} A & \longrightarrow & (A/\mathfrak{p})_\mathfrak{q} \\ \cup\!| & & \cup\!| \\ I & \longrightarrow & \sqrt{I (A/\mathfrak{p})_\mathfrak{q}} = \text{max'l ideal.} \end{array}$$

The Key Lemma $I, J, A, M, \mathfrak{p}, s, d$. Assume

(1) $\omega_A^\bullet$ exists
(2) $\mathfrak{p} \notin V(I) \cap V(J)$
(3) $cd(A, I) \le d$
(4) $\forall \mathfrak{p}' \subseteq \mathfrak{p}$, $\forall \mathfrak{q} \in V(\mathfrak{p}') \cap V(I) \cap V(J)$, $\varphi_M(\mathfrak{p}', \mathfrak{q}) > d+s$.

Then $\exists f \in A$, $f \notin \mathfrak{p}$ which kills $H^i(R\Gamma_J(M)^\wedge_I)$ for $i < s$.

Recall: $\forall \mathfrak{p} \subseteq \mathfrak{q}$.
$\varphi_M(\mathfrak{p}, \mathfrak{q}) = \operatorname{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim (A/\mathfrak{p})_\mathfrak{q}$.

pf. Since $R\Gamma_J(M)^\wedge_I \underset{\uparrow}{=} (R\Gamma_I(R\Gamma_J(M)))^\wedge_I = (R\Gamma_{I+J}(M))^\wedge_I$.

$\{(R\Gamma_I(K))^\wedge_I = K^\wedge_I\}$

We may replace J by $I+J$.

Then $R\Gamma_J(M)^\wedge_I = R\mathrm{Hom}_A(\underbrace{R\Gamma_I(A)}_{0,\dots,d}, R\Gamma_J(M))$.

$\xRightarrow{S.S.}$ Enough to kill $H^i_J(M)$, for $i \le s+d$.

Check conditions for Faltings Ann. Thm:

$\mathfrak{p} \notin T = V(J) \subseteq T' = \operatorname{Spec}(A) \setminus \{\text{pts which specialize to } \mathfrak{p}\}$.

Then Faltings gives J' w/ $V(J') \subseteq T' \not\ni \mathfrak{p}$ and J' kills $H^i_J(M)$, $i \le s+d$.

$\mathfrak{p} \notin V(J') \Longleftrightarrow J' \not\subseteq \mathfrak{p}$ ~~hence~~ $\Longleftrightarrow \exists f \in J'$, $f \notin \mathfrak{p}$.

Tags in SP. Easier case: 0DXK.
General case: 0EFH

Fix $A, I, J, M, \omega_A^\bullet, s, d$. $\varphi_M(\mathfrak{p}, \mathfrak{q}) = \operatorname{depth}(M_\mathfrak{p}) + \dim((A/\mathfrak{p})_\mathfrak{q})$

$T = \{\mathfrak{p} \subseteq A \mid V(\mathfrak{p}) \cap V(I) \subseteq V(J) \cap V(I)\} \subseteq \operatorname{Spec}(A)$, stable under specialization

Helper Lemma Assume

(3) $\mathfrak{p} \notin V(I)$, $\mathfrak{p} \in T \Rightarrow \dim((A/\mathfrak{p})_\mathfrak{q}) \le d$ for some $\mathfrak{q} \in V(\mathfrak{p}) \cap V(J) \cap V(I)$.
(4) $\mathfrak{p} \notin V(I)$, $\mathfrak{p} \notin T \Rightarrow$ either $\operatorname{depth}(M_\mathfrak{p}) \ge s$ OR $\varphi_M(\mathfrak{p}, \mathfrak{q}) > s+d$ for all $\mathfrak{q} \in V(\mathfrak{p}) \cap V(J) \cap V(I)$.

Then $\exists J_0 \subseteq J$ w/ $V(J_0) \cap V(I) = V(J) \cap V(I)$ s.t.

$$H^i_T(M) = H^i_{J_0}(M), \quad i \le s.$$

and the modules are annihilated by some power of $J_0 I$.

pf. Annihilation got from Faltings by considering $T \subseteq T' = T \cup V(I)$.
To find J_0, you look at $\overline{\{\mathfrak{p} \notin V(I), \mathfrak{p} \in T, \operatorname{depth}(M_\mathfrak{p}) \le s\}} \underset{\text{this}}{\overset{\text{show}}{\subseteq}} T$.

Rmk In SP, he proves that for all $J_0' \subseteq J_0$ w/ $V(J_0') \cap V(I) = V(J) \cap V(I)$
then $H^i_{J_0}(M) \xrightarrow{\cong} H^i_{J_0'}(M)$.

Tags in SP. Easier case: 0DXM
General case: 0EFI

Main Result Assume (1) A is I-adically cplt (2) $\mathfrak{p} \in V(I)$: no condition
(3) $cd(A, I) \le d$ (4) $\mathfrak{p} \notin V(I)$, $\mathfrak{p} \notin T$, then $\operatorname{depth}(M_\mathfrak{p}) \ge s$ OR $\varphi_M(\mathfrak{p}, \mathfrak{q}) > s+d$ $\forall \mathfrak{q} \in V(\mathfrak{p}) \cap V(J) \cap V(I)$.
(5) if $\mathfrak{p} \notin V(I)$, $\mathfrak{p} \notin T$, $V(\mathfrak{p}) \cap V(J) \cap V(I) \neq \emptyset$ and $\operatorname{depth}(M_\mathfrak{p}) < s$.
Then at least one of the following is true:
(5a) $\dim(\operatorname{Supp}(M_\mathfrak{p})) < s+2$ (OK, if M has (S_s) away from $V(I) \cup T$).
(5b) $\delta(\mathfrak{p}) > d + \delta_{max} - 1$
(5c) $\varphi_M(\mathfrak{p}, \mathfrak{q}) > d + s + \delta_{max} - \delta_{min} - 2$, $\forall \mathfrak{q} \in V(\mathfrak{p}) \cap V(J) \cap V(I)$.
Here δ_{max} & δ_{min} are max/min of δ on $V(J) \cap V(I)$.

Then with J_0 as in the helper Lemma (which holds under our conditions)

$$H^i_{J_0}(M) \xrightarrow{\cong} H^i(R\Gamma_J(M)^\wedge_I) \quad \text{for } i \le s.$$

(i.e., $(**)$ is an isom for $i \le s$).

Tags in SP: Easier case: 0DXP
General case: 0EFL.

Special case: A = local ring, $I \subseteq \mathfrak{m} = J$.
$\boxtimes$ condition (5) is empty (... ? ...)
$\boxtimes$ $H^i(R\Gamma_{\mathfrak{m}}(M)^\wedge_I) = \lim_n H^i_{\mathfrak{m}}(M/I^nM)$

Thm. $X = \operatorname{Spec}(A) \supseteq U = X \setminus \{\mathfrak{m}\}$, $Y = V(I)$, $\mathcal{F}$ coherent on U.
Assume: A is I-adically complete, has $\omega_A^\bullet$
(1) $\operatorname{cd}(A, I) \le d \Rightarrow \underset{\text{coherent cohomological dimension}}{\text{coh. coh. dim}} (X \setminus Y) \le d-1$
(2) $x \in X \setminus Y$, then $\operatorname{depth}(\mathcal{F}_x) \ge s$ OR $\operatorname{depth}(\mathcal{F}_x) + \dim \overline{\{x\}} > s+d$. (in X)
Then $\operatorname{colim}_{Y \cap U \subseteq V \underset{\text{open}}{\subseteq} U} H^i(V, \mathcal{F}) \xrightarrow{\cong} \lim_n H^i(U, \mathcal{F}/I^n\mathcal{F})$ for $i < s$.

Example $Z \subseteq \mathbb{P}^n_k$ closed subscheme, $\mathcal{F}$ coherent on $\mathbb{P}^n$. ~~Then~~ If c.c.d.$(\mathbb{P}^n \setminus Z) \le d-1$,
$\forall\, t \in \mathbb{P}^n \setminus Z$ have $\operatorname{depth}(\mathcal{F}_t) \ge s$ OR $\operatorname{depth}(\mathcal{F}_t) + \dim(\overline{\{t\}}) \ge d+s$.
Then for all $e \in \mathbb{Z}$ we have

$$\operatorname{colim}_{Z \subseteq V \underset{\text{open}}{\subseteq} \mathbb{P}^n_k} H^i(V, \mathcal{F}(e)) = \lim_r H^i(Z_r, \mathcal{F}(e)|_{Z_r}) \quad \text{for } i < s.$$

Extreme case I: $\mathcal{F} = \mathcal{O}_{\mathbb{P}^n}$, Z = hypersurface. Pick $d=1$, $s = n-1$.

$$H^i\left(\mathbb{P}^n \setminus \left\{\begin{smallmatrix}\text{finite \# pts} \\ \text{not on } Z\end{smallmatrix}\right\}, \mathcal{O}(e)\right) \xrightarrow{\cong} \lim_r H^i(Z_r, \mathcal{O}_{Z_r}(e)) \text{ for } i < n-1.$$

Extreme case II: $\mathcal{F} = \mathcal{O}_{\mathbb{P}^n}$, coherent cohomological dimension of $\mathbb{P}^n \setminus Z \le n-2$.
Choose $d = n-1$, $s = 1$. $H^0(V, \mathcal{O}_V) \cong \lim H^0(Z_r, \mathcal{O}_{Z_r}) \Rightarrow Z$ is conn'd.

Example: $\mathbb{P}^3 \setminus L_1 \sqcup L_2$ for disjoint L_1 & L_2 lines.
Then its c.c.d. $= 2$.

In the thm before, we have extra:

Extra 1. If $\operatorname{depth}(\mathcal{F}_x) + \dim(\overline{\{x\}}) > s$ for all $x \in U \cap Y$, then ~~they are~~ for $i < s$ we have: $\operatorname{colim} H^i(V, \mathcal{F}) \xrightarrow{\cong} \lim_n H^i(U, \mathcal{F}/I^n\mathcal{F})$ are finite A-mod

Extra 2. If $\operatorname{depth}(\mathcal{F}_x) > s$ for $x \in X \setminus Y$ s.t. $\overline{\{x\}} \cap Y = \{\mathfrak{m}\}$, then

$$H^i(U, \mathcal{F}) \xrightarrow{\cong} \lim_n H^i(U, \mathcal{F}/I^n\mathcal{F}).$$

~~Applicati~~

Application I.

$s=1$. Our condition is $\quad (d = cd(A,I))$.
$x \in Ass(\mathcal{F}) \Rightarrow \dim(\overline{\{x\}}) > cd(A,I)+1$.

For example, if A is a domain & $\mathcal{F} = \mathcal{O}_X$, then we're saying
$$\dim(A) > cd(A,I)+1$$
(This condition implies $\dim Y \geq 2$). The thm says
$$H^0(V, \mathcal{O}_V) \xrightarrow{\sim} \varprojlim H^0(U \cap Y_n, \mathcal{O}_{Y_n}).$$
Hence $Y \setminus \{m\}$ is conn'd.

Cor. If $Y \subseteq X = \mathrm{Spec}$ (Noeth. local domain, cplt w.r.t. I) and if coh.coh.dim$(X \setminus Y) + 2 < \dim X$. Then $Y \setminus \{m\}$ is conn'd.

Proj. version $\quad X$ proj. vty, $Z \subseteq X$ closed subvty.
coh.coh.dim.$(X \setminus Z) < \dim(X) - 1$,
then Z is connected.

Optimal conn'dness (local case) Result: A cplt local ring (cplt w.r.t. m). d = min. dim. of irred. of X.
c = min. dim. of $Z \subseteq X$ s.t. $X \setminus Z$ disconn'd.
$I \subseteq A$. Then for $T \subseteq Y = V(I)$ closed, if
$$\dim(T) < \min(c, d-1) - cd(A,I) \Rightarrow Y \setminus T \text{ conn'd}.$$
In particular, $U \cap Y$ is conn'd if $cd(A,I) < \min(c, d-1)$.

Special case: A = cplt domain, $d = \dim A$, $(c = d)$.
$I = (f_1, \cdots, f_r)$ w/ $r < d-1$.
Then $V(f_1, \cdots, f_r) \setminus \{m\}$ is conn'd.

E.g. $X \subseteq \mathbb{P}^n$ closed subvty. $r < \dim X \Rightarrow X \cap H_1 \cap \cdots \cap H_r$ is conn'd.
where $H_i \subseteq \mathbb{P}^n$ hypersurface.

Fundamental gps.

Lemma $\quad$ w/ $(A,m), I$ as in $(*)$. the functor
$$\mathrm{F\acute{E}t}(U) \longrightarrow \mathrm{F\acute{E}t}(U \cap Y).$$
is fully faithful.

$(*)$: • A is I-adically cplt, has ω_A, each irred. component of X meets $U \cap Y$ and each irred. component of X has $\dim > cd(A,I)+1$.
• if $x \in X \setminus Y$ w/ $\overline{\{x\}} \cap Y = \{m\}$, then depth $\mathcal{O}_{X,x} \geq 2$.
[Last condition is automatic if A is (S_2)]

Cor. If purity holds for Y, then it holds for X.

Defn. Let X be a local scheme, we say <u>purity holds for X</u> iff $\mathrm{F\acute{E}t}(X) \longrightarrow \mathrm{F\acute{E}t}(U)$ is essentially surj.

pf of Cor. First reduce to A cplt local. $(\ldots)$.

$$\begin{tikzcd}
 & \mathrm{F\acute{E}t}(A) \arrow[r] \arrow[d] & \mathrm{F\acute{E}t}(U) \arrow[d, "\text{fully faithful}"] \\
\mathrm{F\acute{E}t}(\kappa) \arrow[ur, "\simeq"] \arrow[r, "\simeq"] & \mathrm{F\acute{E}t}(Y) \arrow[r] & \mathrm{F\acute{E}t}(U \cap Y)
\end{tikzcd}$$

pf of Lemma $\begin{array}{c} W_i \\ \downarrow \pi_i \\ U \end{array}$ finite étale $i=1,2$. $\mathcal{F} = \mathcal{H}om_{\mathcal{O}_U}(\pi_{2*}\mathcal{O}_{W_2}, \pi_{1*}\mathcal{O}_{W_1})$
finite locally free $\mathcal{O}_U$-module.

$$\text{if } \begin{tikzcd} W_1|_Y \arrow[rr, "\alpha"] \arrow[dr] & & W_2|_Y \arrow[dl] \\ & Y & \end{tikzcd} \rightsquigarrow \hat{\alpha} = \lim \alpha_n \in \lim H^0(U, \mathcal{F}/I^n\mathcal{F}).$$

By extra 2 of previous thm, $\hat{\alpha}$ comes from elt in $H^0(U, \mathcal{F})$.

Thm (Purity) Any reg. local ring of dim ≥ 2 has purity.

pf. Step 0. reduce to cplt local ring $(\dots)$.

Step 1. case $\dim(A)=2$, b/c A is normal, the statement is eq. to $\left.\begin{array}{l} A \subseteq B \text{ finite, étale cover} \\ \text{~~punctual~~ of punctured Spec, } B \text{ normal} \end{array}\right\} \Rightarrow A \to B$ étale.

$\dim(A)=2 \Rightarrow \dim(B)=2 \Rightarrow B$ is Cohen-Macaulay $\Rightarrow A \xrightarrow[\text{flat}]{} B$ "miracle flatness".

$\Rightarrow B$ free as A-mod. $\Rightarrow \mathrm{disc}_{B/A} \in A$, cut out at most $\{m\}$. $\boxtimes$

The case $\dim(A)>2$: pick $f \in m_A \setminus m_A^2$, set $I=(f)$
$\Rightarrow$ apply Corollary.

More interesting: algebraization of formal coherent modules. This would give $\mathrm{F\acute{E}t}(U) \xrightarrow[\uparrow \text{ essential surj.}]{} \mathrm{F\acute{E}t}(U \cap Y) \cong \mathrm{F\acute{E}t}(U \cap Y_n)$.

Thm (Grothendieck) Any l.c.i. (A, m), $\dim(A) \geq 3$ has purity.

The problem: A = Noetherian ring, $I \subseteq \mathfrak{a} \subseteq A$ ideals.
A is I-adically cplt. $X = \mathrm{Spec}(A) \supseteq Y = V(I) \supseteq Z = V(\mathfrak{a})$.
$U = X \setminus Z$, Y_n = n-th infinitesimal nbhd of Y in X. $\hat{X} = \mathrm{Spf}(A)$.
$\hat{U} =$ "$U \times_X \hat{X}$".

$$\widehat{\{\cdot\}}: \begin{tikzcd} \mathrm{Coh}(\hat{X}) \arrow[r, "\text{restrict}"] \arrow[d, equal] & \mathrm{Coh}(\hat{U}) = \left\{\begin{array}{l} \text{inverse system of } \mathcal{F}_n \text{ of} \\ \text{Coh. } \mathcal{O}_{U_n}\text{-mod., } \mathcal{F}_{n+1}/I^n = \mathcal{F}_n \end{array}\right\} \\ \mathrm{Coh}(X) \arrow[r, "\text{ess. surj.}"] & \mathrm{Coh}(U) \arrow[u] \end{tikzcd}$$

Given $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(X)$, set $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. Then

$$\begin{tikzcd} \mathrm{Hom}_X(\mathcal{F}, \mathcal{G}) = \Gamma(X, \mathcal{H}) \arrow[r] \arrow[d] & \mathrm{Hom}_U(\mathcal{F}, \mathcal{G}) \arrow[d] \\ \mathrm{Hom}_{\hat{U}}(\hat{\mathcal{F}}, \hat{\mathcal{G}}) = \Gamma(\hat{U}, \hat{\mathcal{H}}) & = \lim H^0(U, \mathcal{H}/I^n\mathcal{H}) \end{tikzcd}$$

Upshot: previous results show that $\widehat{\{\cdot\}}$ is fullyfaithful on some subcat of coh. modules on X satisfying some depth conditions.

Q: What about essential surjectivity?

Strategy: Pick $(\mathcal{F}_n) \in \mathrm{Coh}(\hat{U})$. Then we look at $M = \lim H^0(U, \mathcal{F}_n)$.
Try to show M is a finite A-mod. and $\widetilde{M/I^nM}|_U \cong \mathcal{F}_n$.
In any case, $\exists$ canonical maps $\widetilde{M}|_U \to \widetilde{M/I^nM}|_U \to \mathcal{F}_n$.

Lemma. If $\{H^0(U,\mathcal{F}_n)\}_{n\geq 1}$ has ML, then

$$\widetilde{M}|_U \longrightarrow \mathcal{F}_n \text{ is surj. } \forall n.$$

pf. we may check stalks at $y \in Y \setminus Z$ correspond to $\mathfrak{q} \subseteq A$. pick $f \in \mathfrak{a} - \mathfrak{q}$. suffices to show $M_f \longrightarrow H^0(U,\mathcal{F}_n)_f = H^0(D(f),\mathcal{F}_n)$.

by ML:
we can pick $m>n$, s.t. $\mathrm{Im}(M \to H^0(U,\mathcal{F}_n)) = \mathrm{Im}(H^0(U,\mathcal{F}_m) \to H^0(U,\mathcal{F}_n))$.

$\mathrm{coker}(M \to H^0(U,\mathcal{F}_n)) \subseteq H^1(U, \mathrm{Ker}(\mathcal{F}_m \to \mathcal{F}_n))$.

but $H^1(U, q\text{-coh})_f = H^1(D(f), q\text{-coh}) = 0$.

Lemma (SGA2). If $\bigoplus_{n\geq 1} H^1(U, \mathrm{Ker}(\mathcal{F}_{n+1} \to \mathcal{F}_n))$ is a finite graded $S = \bigoplus_{n\geq 0} I^n/I^{n+1}$-mod, then $\{H^0(U,\mathcal{F}_n)\}_{n\geq 1}$ has ML.

pf. $0 \to \underbrace{\mathrm{Ker}(\mathcal{F}_{n+1} \to \mathcal{F}_n)}_{= I^n\mathcal{F}_{n+1}} \to \mathcal{F}_{n+1} \to \mathcal{F}_n \to 0$

$\delta_n : H^0(U,\mathcal{F}_n) \longrightarrow H^1(U, \mathrm{Ker}(\mathcal{F}_{n+1} \to \mathcal{F}_n))$.

$\bigoplus \mathrm{Im}(\delta_n)$ is S-submodule, hence f.g.

Pick N, s_j, $j=1,\dots,N$, $s_j \in H^0(U,\mathcal{F}_{n_j})$ s.t. $\{\delta_{n_j}(s_j)\}$ generate $\bigoplus \mathrm{Im}(\delta_n)$. Then for $n > \max(n_j)$ and $s \in H^0(U,\mathcal{F}_n)$

$\delta_n(s) = \sum_j f_j \delta_{n_j}(s_j)$ some $f_j \in I^{n-n_j}$.

$\to \delta_n(s - \sum f_j s_j) = 0 \Rightarrow s - \sum f_j s_j$ lifts to $s' \in H^0(U,\mathcal{F}_{n+1})$.

$\Rightarrow \mathrm{Im}(H^0(\mathcal{F}_n) \to H^0(\mathcal{F}_{n-\max(n_j)})) = \mathrm{Im}(H^0(\mathcal{F}_{n+1}) \to H^0(\mathcal{F}_{n+1-\max(n_j)}))$.

Improvement: enough to show $\bigoplus_{n\geq 0} \bigcap_{m>n} \mathrm{Im}(H^1(I^n\mathcal{F}_m) \to H^1(I^n\mathcal{F}_{n+1}))$ is finite.

Lemma. If the limit top. on $M = \lim H^0(\mathcal{F}_n)$ is the I-adic top., then $\widetilde{M/I^nM} =: \mathcal{G}_n$ restricted to a coh. module on U & $(\mathcal{G}_n|_U) \to (\mathcal{F}_n)$ is inj in $\mathrm{Coh}(\widehat{U})$.

Rmk. · Situation where the two top. do not coincide (Tag 0EH8).
· If I is a principal ideal, then these 2 top always coincide.

Sketch of pf: Set $N_n = \mathrm{Im}(M \to H^0(U,\mathcal{F}_n))$

easily see: $\widetilde{N_n}|_U \hookrightarrow \mathcal{F}_n \leftarrow$ coherent.

for $n' \gg n$

$$\begin{array}{ccc} & & \widetilde{N_n}|_U \\ & \nearrow & \downarrow \\ \widetilde{M/I^{n'}M}|_U & & \end{array}$$

$\exists$ map of proj. systems the other way, hence $\{\widetilde{N_n}|_U\}_{n\geq 1} = \{\widetilde{M/I^nM}|_U\}_{n\geq 1} \hookrightarrow \{\mathcal{F}_n\}_{n\geq 1}$

Lemma. If $\mathrm{cd}(A,I)=1$, then the limit top. on $H^p = \lim H^p(U,\mathcal{F}_n)$ is the I-adic top.

pf. Special case: Suppose $I=(f)$ and $\widehat{\mathcal{F}} \xrightarrow{f} \widehat{\mathcal{F}}$ inj.
Then $H^p(U,\widehat{\mathcal{F}}) \twoheadrightarrow H^p$.

So now say $\xi \in H^p(U,\widehat{\mathcal{F}})$ maps to zero in $H^p(U,\mathcal{F}_n)$.

Then $0 \to \widehat{\mathcal{F}} \xrightarrow{f^n} \widehat{\mathcal{F}} \to \mathcal{F}_n \to 0$
shows $\xi = f^n\xi'$ for some ξ'.

Conclusion: If ⓐ $\bigoplus H^1(\mathcal{U}, \mathrm{Ker}(\mathcal{F}_{n+1} \to \mathcal{F}_n))$ is a finite $\bigoplus I^n/I^{n+1}$-mod.

ⓑ $\mathrm{cd}(A, I) = 1$

ⓒ for any n, the image $M \to H^0(\mathcal{U}, \mathcal{F}_n)$ is a finite A-mod.

Then $(\mathcal{F}_n) = \widehat{M}|_{\mathcal{U}}$ and M is finite.

pf. To see finiteness of M: $\left.\begin{array}{l} A \text{ is } I\text{-adic cplt} \\ \bigcap_n I^n M = 0 \\ M/IM \text{ finite} \end{array}\right\} \Rightarrow M$ is finite.

the image of $M \to H^0(\mathcal{U}, \mathcal{F}_n)$ maps onto M/IM for $n \gg 0$ due to coincidence of top. (ⓑ) $\Rightarrow$ M/IM finite.

Special case: · $I = (f)$ f nonzerodivisor.

· $\mathcal{F}_n$ locally free $\mathcal{O}_{\mathcal{U}}/f^n\mathcal{O}_{\mathcal{U}}$-mod.

(*) · $H^i(\mathcal{U}, \mathcal{O}_{\mathcal{U}}/f\mathcal{O}_{\mathcal{U}})$ finite A-mod. $i = 0, 1$.

Then $(\mathcal{F}_n) = \widehat{\mathcal{F}}$ for some $\mathcal{F}$ coh. on $\mathcal{U}$ (or on X).

Rmk. If A has dualizing cplx, then (*) is equiv. to

$$\mathrm{depth}((A/f)_{\mathfrak{q}}) + \dim((A/\mathfrak{q})_{\mathfrak{p}}) > 2$$

$$\forall\ \mathfrak{q} \in V(f) \setminus V(\mathfrak{a}) \text{ and } \mathfrak{p} \in V(\mathfrak{q}) \cap V(\mathfrak{a}).$$

e.g. A is Cohen Macaulay, $\forall Z' \subseteq Z$ irred.

$\mathrm{codim}(Z', Y) \geq 3$.

Set up: $X = \mathrm{Spec}\, A \supseteq Y = V(I) \supseteq Z = V(\mathfrak{a})$, $I \subseteq \mathfrak{a} \subseteq A$.

A is I-adically complete. $\mathcal{U} = X \setminus Z$.

$$\mathrm{coh}(X) \twoheadrightarrow \mathrm{Coh}(\mathcal{U}) \to \mathrm{Coh}(\widehat{\mathcal{U}}).$$

Example. $A = k[x,y][[t]] \supseteq \mathfrak{a} = (x, y, t) \supseteq I = (t)$.

$$\widehat{\mathcal{U}} = \underbrace{\mathrm{Spf}(\widehat{A[\tfrac{1}{x}]})}_{\widehat{\mathcal{U}}_1} \cup \underbrace{\mathrm{Spf}(\widehat{A[\tfrac{1}{y}]})}_{\widehat{\mathcal{U}}_2} \text{ glued along } \mathrm{Spf}(\widehat{A[\tfrac{1}{xy}]})$$

Example (a) The closed formal subscheme $\widehat{Z}$ of $\widehat{\mathcal{U}}$ cut out by

$(1) \subseteq \widehat{A[1/x]}$ & $(x - \frac{t}{y}) \subseteq \widehat{A[1/y]}$.

is <u>NOT</u> completion of a $Z \subseteq \mathcal{U}$.

Hence $\mathcal{O}_{\widehat{Z}}$ is <u>NOT</u> algebraizable.

Example (b) The invertible $\mathcal{O}_{\widehat{\mathcal{U}}}$-module $\widehat{\mathcal{L}}$ given by $1 - \frac{t}{xy} \in (\widehat{A[\frac{1}{xy}]})^*$.

is <u>NOT</u> algebraizable; $\widehat{\mathcal{L}}$ has a global section:

$$s = (x \text{ on } \widehat{\mathcal{U}}_1,\ x - \tfrac{t}{y} \text{ on } \widehat{\mathcal{U}}_2)$$

zero's of (s) is $\widehat{Z}$ from example (a).

If $\widehat{\mathcal{L}}$ algebraizes to $\mathcal{F}$ for some ~~torsion~~ free $\mathcal{F}$, (or reflexive).

then $s \in \Gamma(\mathcal{U}, \mathcal{F})$ (by previous results),

then zero's Z of (s) would give algebraization of $\widehat{Z}$.

Example (c): Look at extensions $0 \to \mathcal{O}_{\widehat{\mathcal{U}}} \to \mathcal{E} \to \mathcal{O}_{\widehat{\mathcal{U}}} \to 0$.

The essential surj. comes down to

$$\operatorname*{colim}_{\mathcal{U} \cap Y \subseteq V \subseteq_{\text{open}} \mathcal{U}} H^1(V, \mathcal{O}_V) \longrightarrow H^1(\mathcal{O}_{\widehat{\mathcal{U}}})$$

$$\sqrt{(f,g,t)} = (x,y,t):\quad \frac{A[\frac{1}{fg}]}{A[\frac{1}{f}] + A[\frac{1}{g}]} \qquad \left(\frac{k[x,y,\frac{1}{xy}]}{k[x,y,\frac{1}{x}] + k[x,y,\frac{1}{y}]}\right)[[t]].$$

not big enough.

Example: $A = k[x,y][[s,t]]/(xs-yt)$. $I=(s,t) \subseteq \mathfrak{a} = (x,y,s,t)$

$$\mathcal{F} = \mathcal{O}_{\hat{\mathcal{U}}}.$$

The limit topology on $\lim H^0(\mathcal{U}, \mathcal{O}_\mathcal{U}/I^n\mathcal{O}_\mathcal{U}) = M$ is <u>NOT</u> I-adic, (I-adic topology is strictly finer):

consider $g = \frac{t}{x} = \frac{s}{y} \in H^0(\mathcal{O}_{\hat{\mathcal{U}}}) = M$.

Claim: $g^n \to 0$ in limit topology, but $g^n \notin IM$ for any n.

pf of Claim: define $\mathbb{G}_m$-action: $\begin{matrix} x,y & \text{wt } 1 \\ s,t & \text{wt } 0 \end{matrix}$, $\mathrm{wt}(g) = -1$.

wts on $H^0(\mathcal{O}_\mathcal{U}/I^n\mathcal{O}_\mathcal{U}) \geq -n+1$

$g^n \in H^0(I^n\mathcal{O}_{\hat{\mathcal{U}}})$, if $g^n = sf_1 + tf_2$, then f_1, f_2 are not both zero in $H^0(\mathcal{O}_\mathcal{U}/I^n)$.

hence the weights don't agree.

Notice that $\mathrm{cd}(A, I) = 2$ in this example.

Improving $\hat{\mathcal{F}} = (\mathcal{F}_n)$: want to construct a canonical:

$(\mathcal{F}_n) \longrightarrow (\mathcal{F}_n')$ of inverse systems s.t.

- $\forall n$, $H^0(\mathcal{U}, \mathcal{F}_n') \;\&\; H^1(\mathcal{U}, \mathcal{F}_n')$ finite A-mod.
- $(\mathcal{F}_n')$ as a pro-system is isomorphic to a $(\mathcal{G}_n) = \hat{\mathcal{G}}$ in $\mathrm{Coh}(\hat{\mathcal{U}})$
- the kernel & coker of $\hat{\mathcal{F}} \to \hat{\mathcal{G}}$ are killed by a power of I.

If this is possible, then we can reduce to the case where $\forall n, \exists\, m \geq n$ s.t. $\mathrm{Im}(H^i(\mathcal{U}, \mathcal{F}_m) \to H^i(\mathcal{U}, \mathcal{F}_n))$ is finite for $i = 0, 1$.

Done, if $I = (f)$ (by last time).

Method: $\mathcal{F}_n \longrightarrow \mathcal{F}_n'$ will be constructed by:

- $\delta_Z(y) = 2 \to$ divided by torsion $\mathcal{F}_{n,y}' = \mathcal{F}_{n,y}/H^0_{\{y\}}(\mathcal{F}_{n,y})$.
- $\delta_Z(y) = 1 \to$ Replace by $H^0(\text{punctured Spec}, \mathcal{F}_{n,y})$
- $\delta_Z(y) \geq 3 \to$ leave alone.

Defn: $\delta_Z : Y \longrightarrow \mathbb{Z}_{\geq 0}$ is

- $y \in Z$ then $\delta_Z(y) = 0$.
- if $y \in Y$, $\delta_Z(y)$ is not defined and $y \longleftarrow y' \underset{\text{specialization}}{\overset{\text{Immediate}}{\rightsquigarrow}} y''$ & $\delta_Z(y'')$ defined, then $\delta_Z(y) = \delta_Z(y'') + 1$.

In another words: $\delta_Z(y) = \min\{\#\text{ times of going down} \mid \text{zigzag } y \text{ to } z \in Z\}$

The conditions to impose will be: $y \in Y \setminus Z$, $\hat{\mathcal{F}}_y = \lim \mathcal{F}_{n,y} \circlearrowleft \widehat{\mathcal{O}_{X,y}}$

if $\delta_Z(y) = 1$, $\mathrm{depth}((\hat{\mathcal{F}}_y)_\mathfrak{p}) + \dim(\widehat{\mathcal{O}_{X,y}}/\mathfrak{p}) > 2$

$\delta_Z(y) = 2$, $\quad$ —— " —— > 1

$\forall\, \mathfrak{p} \in \mathrm{Spec}(\widehat{\mathcal{O}_{X,y}})$ and $\mathfrak{p} \notin V(I\widehat{\mathcal{O}_{X,y}})$.

Then OK if $I = (f)$

Example: $I = (f)$, f nonzerodivisor.

- $A[\frac{1}{f}]$ normal.
- $\hat{\mathcal{F}}_y[1/f]$ reflexive, $\forall y \in Y \setminus Z$.
- Y Jacobson, equidim'l.
- $\dim(Z) + 2 < \dim Y$ $[\Leftrightarrow \dim(Z) + 3 < \dim(X)]$.